Momentum picture of motion in Lagrangian quantum field theory

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Short title: **Momentum picture of motion in QFT**

Basic ideas: → June & December, 2001
Began: → January 4, 2002
Ended: → January 6, 2002
Initial typeset: → January 9 – January 16, 2002
Last update: → November 1, 2003
Produced: → February 1, 2008

http://www.arXiv.org e-Print archive No.: hep-th/0311003

BOZHO® TM

Subject Classes:
Quantum field theory

2000 MSC numbers:	***2003 PACS numbers:***
81Q99, 81T99	*03.70.+k, 11.10.Ef,* 11.90.+t, 12.90.+b

Key-Words:
Quantum field theory, Pictures of motion
Pictures of motion in quantum field theory, Momentum picture
Equations of motion, Euler-Lagrange equations, Heisenberg equations/relations

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Contents

Abstract

The basic aspects of the momentum picture of motion in Lagrangian quantum field theory are given. Under some assumptions, this picture is a 4-dimensional analogue of the Schrödinger picture: in it the field operators are constant, spacetime-independent, while the state vectors have a simple, exponential, spacetime-dependence. The role of these assumptions is analyzed. The Euler-Lagrange equations in momentum picture are derived and attention is paid on the conserved operators in it.

1. Introduction

The main item of the present work is a presentation of the basic aspects of the *momentum picture of motion* in Lagrangian quantum field theory, suggested in [1]. In a sense, under some assumptions, this picture is a 4-dimensional analogue of the Schrödinger picture: in it the field operators are constant, spacetime-independent, while the state vectors have a simple, exponential, spacetime-dependence. This state of affairs offers the known merits of the Schrödinger picture (with respect to Heisenberg one in quantum mechanics [2]) in the region of quantum field theory.

We should mention, in this paper it is considered only the Lagrangian (canonical) quantum field theory in which the quantum fields are represented as operators, called field operators, acting on some Hilbert space, which in general is unknown if interacting fields are studied. These operators are supposed to satisfy some equations of motion, from them are constructed conserved quantities satisfying conservation laws, etc. From the view-point of present-day quantum field theory, this approach is only a preliminary stage for more or less rigorous formulation of the theory in which the fields are represented via operator-valued distributions, a fact required even for description of free fields. Moreover, in non-perturbative directions, like constructive and conformal field theories, the main objects are the vacuum mean (expectation) values of the fields and from these are reconstructed the Hilbert space of states and the acting on it fields. Regardless of these facts, the Lagrangian (canonical) quantum field theory is an inherent component of the most of the ways of presentation of quantum field theory adopted explicitly or implicitly in books like [3,4,5,6,7,8,9,10]. Besides the Lagrangian approach is a source of many ideas for other directions of research, like the axiomatic quantum field theory [5, 9, 10].

In Sect. 2 are reviewed some basic moments of the Lagrangian formalism in quantum field theory. In Sect 3 are recalled part of the relations arising from the assumption that the conserved operators are generators of the corresponding invariance transformations of the action integral; in particular the Heisenberg relations between the field operators and momentum operator are written.

The momentum picture of motion is defined in Sect 4. Two basic restrictions on the considered quantum field theories is shown to play a crucial role for the convenience of that picture: the mutual commutativity between the components of the momentum operator and the Heisenberg commutation relation between them and the field operators. If these conditions hold, the field operators in momentum picture become spacetime-independent and the state vectors turn to have exponential spacetime-dependence. In Sect 5, the attention is called to the Euler-Lagrange equations and dynamical variables in momentum picture. In Sect. 6 is given an idea of the momentum representation in momentum picture and the similarity with that representation in Heisenberg picture is pointed. In Sect. 7 is made a comparison between the momentum picture in qunatum field theory and the Schrödinger pinctur in quantum mechanics. Some closing remarks are given in Sect. 8. It is pointed that the above-mentioned restrictions are fundamental enough to be put in the basic postulates of quantum field theory, which may result in a new way of its (Lagrangian) construction.

The books [3, 5, 4] will be used as standard reference works on quantum field theory. Of course, this is more or less a random selection between the great number of (text)books and papers on the theme to which the reader is referred for more details or other points of view. For this end, e.g., [11, 6] or the literature cited in [3, 5, 4, 11, 6] may be helpful.

Throughout this paper $\hbar$ denotes the Planck's constant (divided by 2π), c is the velocity of light in vacuum, and i stands for the imaginary unit. The superscript $\dagger$ means Hermitian conjugation (of operators or matrices), and the symbol $\circ$ denotes compositions of mappings/operators.

The Minkowski spacetime is denoted by M. The Greek indices run from 0 to $\dim M = 4$. All Greek indices will be raised and lowered by means of the standard 4-dimensional Lorentz metric tensor $\eta^{\mu\nu}$ and its inverse $\eta_{\mu\nu}$ with signature $(+\,-\,-\,-)$. The Einstein's summation convention over indices repeated on different levels is assumed over the whole range of their values.

2. Lagrangian formalism

Let us consider a system of quantum fields, represented in Heisenberg picture of motion by field operators $\tilde{\varphi}_i(x)\colon \mathcal{F} \to \mathcal{F}$, with $i = 1, \dots, n \in \mathbb{N}$, in system's Hilbert space $\mathcal{F}$ of states and depending on a point x in Minkowski spacetime M. Here and henceforth, all quantities in Heisenberg picture, in which the state vectors are spacetime-independent contrary to the field operators and observables, will be marked by a tilde (wave) "˜" over their kernel symbols. Let

$$\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\tilde{\varphi}_i(x), \partial_\nu \tilde{\varphi}_j(x)) \tag{2.1}$$

be the system's Lagrangian, which is supposed to depend on the field operators and their first partial derivatives.[1] We expect that this dependence is polynomial or in a form of convergent power series, which can be treated term by term. The Euler-Lagrange equations for the Lagrangian (2.1), i.e.

$$\frac{\partial \tilde{\mathcal{L}}(\tilde{\varphi}_j(x), \partial_\nu \tilde{\varphi}_l(x))}{\partial \tilde{\varphi}_i(x)} - \frac{\partial}{\partial x^\mu} \frac{\partial \tilde{\mathcal{L}}(\tilde{\varphi}_j(x), \partial_\nu \tilde{\varphi}_l(x))}{\partial(\partial_\mu \tilde{\varphi}_i(x))} = 0, \tag{2.2}$$

are identified with the field equations (of motion) for the quantum fields $\tilde{\varphi}_i(x)$.[2]

For definiteness, above and below, we consider a quantum field theory *before* normal ordering and, possibly, without (anti)commutation relations (see Sect. 4). However, our investigation is, practically, independent of these procedures and can easily be modified to include them.

Following the standard procedure [3,4,11,5] (see also [12]), from the Lagrangian (2.1) can be constructed the densities of the conserved quantities of the system, viz. the energy-momentum tensor $\tilde{\mathcal{T}}_{\mu\nu}(x)$, charge current $\tilde{\mathcal{J}}_\mu(x)$, the (total) angular momentum density operator

$$\tilde{\mathcal{M}}^\lambda_{\mu\nu} = \tilde{\mathcal{L}}^\lambda_{\mu\nu}(x) + \tilde{\mathcal{S}}^\lambda_{\mu\nu}(x), \tag{2.3}$$

where

$$\tilde{\mathcal{L}}^\lambda_{\mu\nu}(x) = x_\mu \tilde{\mathcal{T}}^\lambda{}_\nu(x) - x_\nu \tilde{\mathcal{T}}^\lambda{}_\mu(x) \tag{2.4}$$

and $\tilde{\mathcal{S}}^\lambda_{\mu\nu}(x)$ are respectively the orbital and spin angular momentum density operators, and others, if such ones exist. The corresponding to these quantities integral ones, viz. the momentum, charge, (total) angular momentum, orbital and spin angular momentum operators,

[1] One can easily generalize the below presented material for Lagrangians depending on higher order derivatives.

[2] In (2.2) and similar expressions appearing further, the derivatives of functions of operators with respect to operator arguments are calculated in the same way as if the operators were ordinary (classical) fields/functions, only the order of the arguments should not be changed. This is a silently accepted practice in the literature [3, 5, 4]. In the most cases such a procedure is harmless, but it leads to the problem of non-unique definitions of the quantum analogues of the classical conserved quantities, like the energy-momentum and charge operators. For some details on this range of problems in quantum field theory, see [12]. In *loc. cit.* is demonstrated that these problems can be eliminated by changing the rules of differentiation with respect to *not*-commuting variables. The paper [12] contains an example of a Lagrangian (describing spin $\frac{1}{2}$ field) whose field equations are *not* the Euler-Lagrange equations (2.2) obtained as just described, but we shall not investigate such cases in the present work.

are respectively defined by:

$$\tilde{\mathcal{P}}_\mu := \frac{1}{c} \int\limits_{x^0=\text{const}} \tilde{\mathcal{T}}_{0\mu}(x)\,\mathrm{d}^3\boldsymbol{x}. \tag{2.5}$$

$$\tilde{\mathcal{Q}} := \frac{1}{c} \int\limits_{x^0=\text{const}} \tilde{\mathcal{J}}_0(x)\,\mathrm{d}^3\boldsymbol{x} \tag{2.6}$$

$$\tilde{\mathcal{M}}_{\mu\nu} = \tilde{\mathcal{L}}_{\mu\nu}(x) + \tilde{\mathcal{S}}_{\mu\nu}(x), \tag{2.7}$$

$$\tilde{\mathcal{L}}_{\mu\nu}(x) := \frac{1}{c} \int\limits_{x^0=\text{const}} \{x_\mu\,\tilde{\mathcal{T}}^0{}_\nu(x) - x_\nu\,\tilde{\mathcal{T}}^0{}_\mu(x)\}\,\mathrm{d}^3\boldsymbol{x} \tag{2.8a}$$

$$\tilde{\mathcal{S}}_{\mu\nu}(x) := \frac{1}{c} \int\limits_{x^0=\text{const}} \tilde{\mathcal{S}}^0_{\mu\nu}(x)\,\mathrm{d}^3\boldsymbol{x} \tag{2.8b}$$

and satisfy the conservation laws

$$\frac{\mathrm{d}\,\tilde{\mathcal{P}}_\mu}{\mathrm{d}x^0} = 0 \quad \frac{\mathrm{d}\,\tilde{\mathcal{Q}}}{\mathrm{d}x^0} = 0 \quad \frac{\mathrm{d}\,\tilde{\mathcal{M}}_{\mu\nu}}{\mathrm{d}x^0} = 0 \tag{2.9}$$

which, in view of (2.5)–(2.8), are equivalent to

$$\partial_\lambda\,\tilde{\mathcal{P}}_\mu = 0 \quad \partial_\lambda\,\tilde{\mathcal{Q}} = 0 \quad \partial_\lambda\,\tilde{\mathcal{M}}_{\mu\nu} = 0 \tag{2.10}$$

and also to

$$\partial^\lambda\,\tilde{\mathcal{T}}_{\lambda\mu} = 0 \quad \partial^\lambda\,\tilde{\mathcal{J}}_\lambda = 0 \quad \partial_\lambda\,\tilde{\mathcal{M}}^\lambda_{\mu\nu} = 0. \tag{2.11}$$

The Lagrangian, as well as the conserved quantities and there densities, are Hermitian operators; in particular, such is the momentum operator,

$$\tilde{\mathcal{P}}^\dagger_\mu = \tilde{\mathcal{P}}_\mu. \tag{2.12}$$

The reader can find further details on the Lagrangian formalism in, e.g., [3, 4, 11, 5, 12].

3. Heisenberg relations

The conserved quantities (2.5)–(2.7) are often identified with the generators of the corresponding transformations, under which the action operator is invariant [3, 11, 13, 5]. This leads to a number of commutation relations between the conserved operators and between them and the field operators. The relations of the latter set are often referred as the Heisenberg relations or equations. Part of them are briefly reviewed below; for details, see *loc. cit.*

The consideration of $\tilde{\mathcal{P}}_\mu$, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{M}}_{\mu\nu}$ as generators of translations, constant phase transformations and 4-rotations, respectively, leads to the following relations:

$$[\tilde{\varphi}_i(x),\,\tilde{\mathcal{P}}_\mu]_- = \mathrm{i}\hbar\frac{\partial\,\tilde{\varphi}_i(x)}{\partial x^\mu} \tag{3.1}$$

$$[\tilde{\varphi}_i(x),\,\tilde{\mathcal{Q}}]_- = \varepsilon(\,\tilde{\varphi}_i)q_i\,\tilde{\varphi}_i(x) \tag{3.2}$$

$$[\tilde{\varphi}_i(x),\,\tilde{\mathcal{M}}_{\mu\nu}]_- = \mathrm{i}\hbar\{x_\mu\partial_\nu\,\tilde{\varphi}_i(x) - x_\nu\partial_\mu\,\tilde{\varphi}_i(x) + I^j_{i\mu\nu}\,\tilde{\varphi}_j(x)\}. \tag{3.3}$$

Here: $q_i = \text{const}$ is the charge of the i^{th} field, $\varepsilon(\tilde{\varphi}_i) = 0$ if $\tilde{\varphi}_i^\dagger = \tilde{\varphi}_i$, $\varepsilon(\tilde{\varphi}_i) = \pm 1$ if $\tilde{\varphi}_i^\dagger \neq \tilde{\varphi}_i$ with $\varepsilon(\tilde{\varphi}_i) + \varepsilon(\tilde{\varphi}_i^\dagger) = 0$, and the constants $I^j_{i\mu\nu} = -I^j_{i\nu\mu}$ characterize the transformation properties of the field operators under 4-rotations. (It is a convention whether to put $\varepsilon(\tilde{\varphi}_i) = +1$ or $\varepsilon(\tilde{\varphi}_i) = -1$ for a fixed i.) Besides, the operators $\tilde{\mathcal{P}}_\mu$, $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{M}}_{\mu\nu}$ satisfy certain commutation relation between themselves, from which we shall write the following two:

$$[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_} = 0 \tag{3.4}$$

$$[\tilde{\mathcal{Q}}, \tilde{\mathcal{P}}_\mu]_{\_} = 0. \tag{3.5}$$

It should be clearly understood, the equations (3.1)–(3.5) are from pure geometrical origin and are completely external to the Lagrangian formalism. However, there are strong evidences that they should hold in a realistic Lagrangian quantum field theory (see [13, § 68] and [3, § 5.3 and § 9.4]). Moreover, (most of) the above relations happen to be valid for Lagrangians that are frequently used, e.g. for the ones describing free fields [13].

4. The momentum picture of motion

Let $\tilde{\mathcal{P}}_\mu$ be the system's momentum operator, given by equation (2.5). Since $\tilde{\mathcal{P}}_\mu$ is Hermitian (see (2.12)), the operator

$$\mathcal{U}(x, x_0) = \exp\Big(\frac{1}{\mathrm{i}\hbar} \sum_\mu (x^\mu - x_0^\mu)\, \tilde{\mathcal{P}}_\mu\Big), \tag{4.1}$$

where $x_0 \in M$ is arbitrarily fixed and $x \in M$, is unitary, i.e.

$$\mathcal{U}^\dagger(x_0, x) := (\mathcal{U}(x, x_0))^\dagger = (\mathcal{U}(x, x_0))^{-1} =: \mathcal{U}^{-1}(x, x_0). \tag{4.2}$$

Let $\tilde{\mathcal{X}} \in \mathcal{F}$ be a state vector in the system's Hilbert space $\mathcal{F}$ and $\tilde{\mathcal{A}}(x) \colon \mathcal{F} \to \mathcal{F}$ be an operator on it. The transformations

$$\tilde{\mathcal{X}} \mapsto \mathcal{X}(x) = \mathcal{U}(x, x_0)(\tilde{\mathcal{X}}) \tag{4.3}$$

$$\tilde{\mathcal{A}}(x) \mapsto \mathcal{A}(x) = \mathcal{U}(x, x_0) \circ (\tilde{\mathcal{A}}(x)) \circ \mathcal{U}^{-1}(x, x_0), \tag{4.4}$$

evidently, preserve the Hermitian scalar product $\langle\cdot|\cdot\rangle \colon \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ of $\mathcal{F}$ and the mean values of the operators, i.e.

$$\langle \tilde{\mathcal{X}} | \tilde{\mathcal{A}}(x)(\tilde{\mathcal{Y}})\rangle = \langle \mathcal{X}(x) | \mathcal{A}(x)(\mathcal{Y}(x))\rangle \tag{4.5}$$

for any $\tilde{\mathcal{X}}, \tilde{\mathcal{Y}} \in \mathcal{F}$ and $\tilde{\mathcal{A}}(x) \colon \mathcal{F} \to \mathcal{F}$. Since the physically predictable/measurable results of the theory are expressible via scalar products in $\mathcal{F}$ [3, 4, 11], the last equality implies that the theory's description via vectors and operators like $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{A}}(x)$ above is completely equivalent to the one via the vectors $\mathcal{X}(x)$ and operators $\tilde{\mathcal{A}}(x)$, respectively. The description of quantum field theory via $\mathcal{X}$ and $\mathcal{A}(x)$ will be called the *momentum picture (of motion (of quantum field theory))* [1].

However, without further assumptions, this picture turns to be rather complicated. The mathematical cause for this is that derivatives of different operators are often met in the theory and, as a consequence of (4.4), they transform as

$$\partial_\mu \tilde{\mathcal{A}}(x) \mapsto \mathcal{U}(x, x_0) \circ (\partial_\mu \tilde{\mathcal{A}}(x)) \circ \mathcal{U}^{-1}(x, x_0) = \partial_\mu \mathcal{A}(x) + [\mathcal{A}(x), \mathcal{H}_\mu(x, x_0)]_{\_} \tag{4.6}$$

$$\mathcal{H}_\mu(x, x_0) := \big(\partial_\mu \mathcal{U}(x, x_0)\big) \circ \mathcal{U}^{-1}(x, x_0) \tag{4.7}$$

from Heisenberg to momentum picture. Here $[\mathcal{A}, \mathcal{B}]_{\_} := \mathcal{A} \circ \mathcal{B} - \mathcal{B} \circ \mathcal{A}$ is the commutator of $\mathcal{A}, \mathcal{B} \colon \mathcal{F} \to \mathcal{F}$. The entering in (4.6), via (4.7), derivatives of the operator (4.1) can be represented as the convergent power series

$$\partial_\mu \mathcal{U}(x, x_0) = \frac{1}{\mathrm{i}\hbar} \mathcal{P}_\mu + \frac{1}{\mathrm{i}\hbar} \sum_{n=1}^{\infty} \frac{1}{(\mathrm{i}\hbar)^n} \frac{1}{(n+1)!} \sum_{m=0}^{n} \big((x^\lambda - x_0^\lambda)\, \tilde{\mathcal{P}}_\lambda\big)^m \circ \tilde{\mathcal{P}}_\mu \circ \big((x^\lambda - x_0^\lambda)\, \tilde{\mathcal{P}}_\lambda\big)^{n-m},$$

where $(\cdots)^n := (\cdots) \circ \cdots \circ (\cdots)$ (n-times) and $(\cdots)^0 := \mathsf{id}_{\mathcal{F}}$ is the identity mapping of $\mathcal{F}$, which cannot be written in a closed form unless the commutator $[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_}$ has 'sufficiently simple' form. In particular, the relation (3.4) entails $\partial_\mu \mathcal{U}(x, x_0) = \frac{1}{\mathrm{i}\hbar} \tilde{\mathcal{P}}_\mu \circ \mathcal{U}(x, x_0)$, so that (4.7) and (4.6) take respectively the form

$$\mathcal{H}_\mu(x, x_0) = \frac{1}{\mathrm{i}\hbar} \tilde{\mathcal{P}}_\mu \tag{4.8}$$

$$\partial_\mu \tilde{\mathcal{A}}(x) \mapsto \partial_\mu \mathcal{A}(x) + \frac{1}{\mathrm{i}\hbar}[\mathcal{A}(x), \tilde{\mathcal{P}}_\mu]_{\_}. \tag{4.9}$$

Notice, the equality (4.8) is possible iff and only if $\mathcal{U}_\mu(x, x_0)$ is a solution of the initial-value problem (see (4.7))

$$\mathrm{i}\hbar \frac{\partial \mathcal{U}(x, x_0)}{\partial x^\mu} = \tilde{\mathcal{P}}_\mu \circ \mathcal{U}(x, x_0) \tag{4.10a}$$

$$\mathcal{U}(x_0, x_0) = \mathsf{id}_{\mathcal{F}}, \tag{4.10b}$$

the integrability conditions for which are exactly (3.4).[3] Since (3.4) and (4.1) imply

$$[\mathcal{U}(x, x_0), \tilde{\mathcal{P}}_\mu]_{\_} = 0, \tag{4.11}$$

by virtue of (4.4), we have

$$\mathcal{P}_\mu = \tilde{\mathcal{P}}_\mu, \tag{4.12}$$

i.e. the momentum operators in Heisenberg and momentum pictures coincide, provided (3.1) holds.

It is worth to be mentioned, equation (4.12) is a special case of

$$\mathcal{A}(x) = \tilde{\mathcal{A}}(x) + [\mathcal{U}(x, x_0), \tilde{\mathcal{A}}(x)]_{\_} \circ \mathcal{U}^{-1}(x, x_0), \tag{4.13}$$

which is a consequence of (4.4) and is quite useful if one knows explicitly the commutator $[\mathcal{U}(x, x_0), \tilde{\mathcal{A}}(x)]_{\_}$. In particular, if

$$\big[[\tilde{\mathcal{A}}(x), \tilde{\mathcal{P}}_\mu]_{\_}, \tilde{\mathcal{P}}_\nu\big]_{\_} = 0 \tag{4.14}$$

and (3.4) holds, then, by expanding (4.1) into a power series, one can prove that

$$[\tilde{\mathcal{A}}(x), \mathcal{U}(x, x_0)]_{\_} = \frac{1}{\mathrm{i}\hbar}(x^\lambda - x_0^\lambda)[\tilde{\mathcal{A}}(x), \tilde{\mathcal{P}}_\lambda]_{\_} \circ \mathcal{U}(x, x_0). \tag{4.15}$$

So, in this case, (4.13) reduces to

$$\mathcal{A}(x) = \tilde{\mathcal{A}}(x) - \frac{1}{\mathrm{i}\hbar}(x^\lambda - x_0^\lambda)[\tilde{\mathcal{A}}(x), \tilde{\mathcal{P}}_\lambda]_{\_}. \tag{4.16}$$

This formula allows to be found an operator in momentum picture if its commutator(s) with (the components of) the momentum operator is (are) explicitly known, provided (3.4) and (4.14) hold. The choice $\tilde{\mathcal{A}}(x) = \tilde{\mathcal{P}}_\mu$ reduces (4.16) to (4.12).

Of course, a transition from one picture of motion to other one is justified if there are some merits from this step; for instance, if some (mathematical) simplification, new physical

[3] For a system with a non-conserved momentum operator $\tilde{\mathcal{P}}_\mu(x)$ the operator $\mathcal{U}(x, x_0)$ should be defined as the solution of (4.10), with $\tilde{\mathcal{P}}_\mu(x)$ for $\tilde{\mathcal{P}}_\mu$, instead of by (4.1); in this case, equation (3.4) should be replace with

$$[\tilde{\mathcal{P}}_\mu(x), \tilde{\mathcal{P}}_\nu(x)]_{\_} + \partial_\nu \tilde{\mathcal{P}}_\mu(x) - \partial_\mu \tilde{\mathcal{P}}_\nu(x) = 0.$$

Most of the material in the present section remains valid in that more general situation.

interpretation etc. occur in the new picture. A classical example of this kind is the transition between Schrödinger and Heisenberg pictures in quantum mechanics [2] or, in a smaller extend, in quantum field theory [3]. Until now we have not present evidences that the momentum picture can bring some merits with respect to, e.g., Heisenberg picture. On the opposite, there was an argument that, without further restrictions, mathematical complications may arise in it.

In this connection, let us consider, as a second possible restriction, a theory in which the Heisenberg equation (3.1) is valid. In momentum picture, it reads

$$[\varphi_i(x),\, \tilde{\mathcal{P}}_\mu - \mathrm{i}\hbar\,\mathcal{H}_\mu]_{\_} = \mathrm{i}\hbar\partial_\mu\varphi_i(x), \tag{4.17}$$

where

$$\tilde{\varphi}_i(x) \mapsto \varphi_i(x) = \mathcal{U}(x,x_0)\circ\tilde{\varphi}_i(x)\circ\mathcal{U}^{-1}(x,x_0). \tag{4.18}$$

are the field operators in momentum picture and the relations (4.4) and (4.6) were applied. The equation (4.17) shows that, if (4.8) holds, which is equivalent to the validity of (3.4), then

$$\partial_\mu\varphi_i(x) = 0, \tag{4.19}$$

i.e. in this case the field operators in momentum picture turn to be constant,

$$\varphi_i(x) = \mathcal{U}(x,x_0)\circ\tilde{\varphi}_i(x)\circ\mathcal{U}^{-1}(x,x_0) = \varphi_i(x_0) = \tilde{\varphi}_i(x_0) =: \varphi_i. \tag{4.20}$$

As a result of the last fact, all functions of the field operators and their derivatives, polynomial or convergent power series in them, become constant operators in momentum picture, which are algebraic functions of the field operators in momentum picture. This is an essentially new moment in the theory that reminds to a similar situation in the Schrödinger picture in quantum mechanics (see [2] and Sect. 7 below).

If $\tilde{\mathcal{P}}_\mu$ is considered, as usual [3,4], as a generator of 4-translations, then the constancy of the field operators in momentum picture is quite natural. In fact, in this case, the transition $\tilde{\varphi}_i(x) \mapsto \varphi_i(x)$, given by (4.18), means that the argument of $\tilde{\varphi}_i(x)$ is shifted by $(x_0 - x)$, i.e. that $\tilde{\varphi}_i(x) \mapsto \varphi_i(x) = \tilde{\varphi}_i(x + (x_0 - x)) = \tilde{\varphi}_i(x_0)$.

Let us turn our attention now to system's state vectors. By definition [3,4], such a vector $\tilde{\mathcal{X}}$ is a spacetime-constant one in Heisenberg picture,

$$\partial_\mu\tilde{\mathcal{X}} = 0. \tag{4.21}$$

In momentum picture, the situation is opposite, as, by virtue of (4.3), the operator (4.1) plays a role of spacetime 'evolution' operator, i.e.

$$\mathcal{X}(x) = \mathcal{U}(x,x_0)(\mathcal{X}(x_0)) = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu)\tilde{\mathcal{P}}_\mu}(\mathcal{X}(x_0)), \tag{4.22}$$

with

$$\mathcal{X}(x_0) = \mathcal{X}(x)|_{x=x_0} = \tilde{\mathcal{X}} \tag{4.23}$$

being considered as initial value of $\mathcal{X}(x)$ at $x = x_0$. Thus, if $\mathcal{X}(x_0) = \tilde{\mathcal{X}}$ is an eigenvector of the momentum operators $\tilde{\mathcal{P}}_\mu = \mathcal{P}_\mu(x)|_{x=x_0}$ $(= \{\mathcal{U}(x,x_0)\circ\tilde{\mathcal{P}}_\mu\circ\mathcal{U}^{-1}(x,x_0)\}|_{x=x_0})$ with eigenvalues p_μ, i.e.

$$\tilde{\mathcal{P}}(\tilde{\mathcal{X}}) = p_\mu\tilde{\mathcal{X}} \qquad \big(= p_\mu\mathcal{X}(x_0) = \mathcal{P}_\mu(x_0)(\tilde{\mathcal{X}}(x_0))\big), \tag{4.24}$$

we have the following *explicit* form of a state vector $\mathcal{X}$:

$$\mathcal{X}(x) = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu)p_\mu}(\mathcal{X}(x_0)). \tag{4.25}$$

It should be understood, this is the *general form of all state vectors in momentum picture*, as they are eigenvectors of all (commuting) observables [5, p. 59], in particular, of the momentum operator.

So, in momentum picture, the state vectors have a relatively simple *global* description. However, their differential (local) behavior is described via a differential equation that may turn to be rather complicated unless some additional conditions are imposed. Indeed, form (4.22), we get

$$\partial_\mu \mathcal{X}(x) = \mathcal{H}_\mu(x, x_0)(\mathcal{X}(x)) \tag{4.26}$$

in which equation the operator $\mathcal{H}_\mu(x, x_0)$ is given by (4.7) and may have a complicated explicit form (*vide supra*). The equality (4.26) has a form similar to the one of the Schrödinger equation, but in '4-dimensions', with '4-dimensional Hamiltonian' $\mathrm{i}\hbar\mathcal{H}_\mu(x, x_0)$. It is intuitively clear, in this context, the operators $\mathrm{i}\hbar\mathcal{H}_\mu(x, x_0)$ should be identified with the components of the momentum operator $\mathcal{P}_\mu$, i.e. the equality (4.8) is a natural one on this background.

Thus, if we accept (4.8), or equivalently (3.4), a state vector $\mathcal{X}(x)$ in momentum picture will be a solution of the initial-value problem

$$\mathrm{i}\hbar\frac{\partial\,\mathcal{X}(x)}{\partial x^\mu} = \mathcal{P}_\mu(\mathcal{X}(x)) \qquad \mathcal{X}(x)|_{x=x_0} = \mathcal{X}(x_0) = \tilde{\mathcal{X}} \tag{4.27}$$

and, respectively, the evolution operator $\mathcal{U}(x, x_0)$ of the state vectors will be a solution of (4.10). Consequently, the equation (3.4) entails not only a simplified description of the operators in momentum picture, but also a natural one of the state vectors in it.

The above discussion reveals that the momentum picture is worth to be employed in quantum field theories in which the conditions

$$[\tilde{\mathcal{P}}_\mu, \tilde{\mathcal{P}}_\nu]_{\_} = 0 \tag{4.28a}$$

$$[\tilde{\varphi}_i(x), \tilde{\mathcal{P}}_\mu]_{\_} = \mathrm{i}\hbar\partial_\mu\,\tilde{\varphi}_i(x) \tag{4.28b}$$

are valid. In that case, the momentum picture can be considered as a 4-dimensional analogue of the Schrödinger picture [1]: the field operators are spacetime-constant and the state vectors are spacetime-dependent and evolve according to the '4-dimensional Schrödinger equation' (4.27) with evolution operator (4.1). More details on that item will be given in Sect. 7 below.

In connection with the conditions (4.28), it should be said that their validity is more a rule than an exception. For instance, in the axiomatic quantum field theory, they hold identically as in this approach, by definition, the momentum operator is identified with the generator of translations [9, 10]. In the Lagrangian formalism, to which (4.28) are external restrictions, the conditions (4.28) seem to hold at least for the investigated free fields and most (all?) interacting ones [13]. For example, the commutativity between the components of the momentum operator, expressed via (4.28a), is a consequence of the (anti)commutation relations and, possibly, the field equations. Besides, it expresses the simultaneous measurability of the components of system's momentum. The Heisenberg relation (4.28b) is verified in [13] for a number of Lagrangians. Moreover, in *loc. cit.* it is regarded as one of the conditions for relativistic covariance in a translation-invariant Lagrangian quantum field theory. All these facts point that the conditions (4.28) are fundamental enough to be incorporated in the basic postulates of quantum field theory, as it is done (more implicitly than explicitly), e.g., in [3, 11, 5]. Some comments on that problem will be presented in Sect. 8 (see also [14, chapter 1]).

5. General aspects of Lagrangian formalism in momentum picture

In this section, some basic moments of the Lagrangian formalism in momentum picture will be considered, provided the equations (4.28) hold.

To begin with, let us recall, in the momentum picture, under the conditions (4.28), the field operators φ_i are constant, i.e. spacetime-independent (which is equivalent to (4.28a)), and the state vectors are spacetime-dependent, their dependence being of exponential type (see (4.22) and (4.25)). As a result of this, one can expect a simplification of the formalism, as it happens to be the case.

Combining (4.6), with $\mathcal{A} = \varphi_i$, (4.8), (4.19) and (4.12), we see that the first partial derivatives of the field operators transform from Heisenberg to momentum picture according to the rule

$$\partial_\mu \tilde{\varphi}_i(x) \mapsto y_{i\mu} := \frac{1}{\mathrm{i}\hbar}[\varphi_i, \mathcal{P}_\mu]_{\_}. \tag{5.1}$$

Therefore the operator ∂_μ, when applied to field operators, transforms into $\frac{1}{\mathrm{i}\hbar}[\cdot, \tilde{\mathcal{P}}_\mu]_{\_} = \frac{1}{\mathrm{i}\hbar}[\cdot, \mathcal{P}_\mu]_{\_}$, which is a differentiation of the operator space over $\mathcal{F}$. An important corollary of (5.1) is that any (finite order) differential expression of $\tilde{\varphi}_i(x)$ transforms in momentum picture into an *algebraic* one of φ_i. In particular, this concerns the Lagrangian (which is supposed to be polynomial or convergent power series in the field operators and their partial derivatives):

$$\begin{aligned}\tilde{\mathcal{L}} \mapsto \mathcal{L} :&= \mathcal{L}(\varphi_i(x)) := \mathcal{U}(x,x_0)\circ \tilde{\mathcal{L}}(\tilde{\varphi}_i(x), \partial_\nu \tilde{\varphi}_j(x)) \circ \mathcal{U}^{-1}(x,x_0)\\ &= \tilde{\mathcal{L}}\big(\mathcal{U}(x,x_0)\circ\tilde{\varphi}_i(x)\circ\mathcal{U}^{-1}(x,x_0), \mathcal{U}(x,x_0)\circ\partial_\nu\tilde{\varphi}_j(x)\circ\mathcal{U}^{-1}(x,x_0)\big)\\ &= \tilde{\mathcal{L}}\big(\varphi_i, \frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\nu]_{\_}\big).\end{aligned}$$

Thus, the Lagrangian (2.1) in momentum picture reads

$$\mathcal{L} = \mathcal{L}(\phi_i) = \tilde{\mathcal{L}}(\varphi_i, y_{j\nu}) \qquad y_{j\nu} = \frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\nu]_{\_}, \tag{5.2}$$

i.e. one has to make simply the replacements $\tilde{\varphi}_i(x) \mapsto \varphi_i$ and $\partial_\nu \tilde{\varphi}_i(x) \mapsto y_{i\nu}$ in (2.1).

Applying the general rule (4.4) to the Euler-Lagrange equations (2.2) and using (4.20) and (5.1), we find, after some simple calculations,[4] the *Euler-Lagrange equations in momentum picture* as

$$\Big\{ \frac{\partial\,\tilde{\mathcal{L}}(\varphi_j, y_{l\nu})}{\partial\varphi_i} - \frac{1}{\mathrm{i}\hbar}\Big[\frac{\partial\,\tilde{\mathcal{L}}(\varphi_j, y_{l\nu})}{y_{i\mu}}, \mathcal{P}_\mu\Big]_{\_}\Big\}\Big|_{y_{j\nu}=\frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\nu]_{\_}} = 0. \tag{5.3}$$

A feature of these equations is that they are *algebraic*, not differential, ones with respect to the field operators φ_i (in momentum picture), provided $\mathcal{P}_\mu$ is regarded as a given known operator. This is a natural fact in view of (4.19).

We shall illustrate the above general considerations on the almost trivial example of a free Hermitian scalar field $\tilde{\varphi}$, described in Heisenberg picture by the Lagrangian $\tilde{\mathcal{L}} = -\frac{1}{2}m^2c^4\,\tilde{\varphi}\circ\tilde{\varphi} + c^2\hbar^2(\partial_\mu\tilde{\varphi})\circ(\partial^\mu\tilde{\varphi}) = \tilde{\mathcal{L}}(\tilde{\varphi}, \tilde{y}_\nu)$, with $m = \text{const}$ and $\tilde{y}_\nu = \partial_\nu\tilde{\varphi}$, and satisfying the Klein-Gordon equation $(\tilde{\Box} + \frac{m^2c^2}{\hbar^2}\,\mathsf{id}_{\mathcal{F}})\,\tilde{\varphi} = 0$, $\tilde{\Box} := \partial_\mu\partial^\mu$. In momentum picture $\tilde{\varphi}$ transforms into the constant operator

$$\varphi(x) = \mathcal{U}(x,x_0)\circ\tilde{\varphi}\circ\mathcal{U}^{-1}(x,x_0) = \varphi(x_0) = \tilde{\varphi}(x_0) =: \varphi \tag{5.4}$$

[4] For details, see [1].

which, in view of (5.3), $\frac{\partial \tilde{\mathcal{L}}}{\partial \varphi} = -m^2c^4\varphi$, and $\frac{\partial \tilde{\mathcal{L}}}{\partial y_\nu} = c^2\hbar^2 y_\mu \eta^{\mu\nu}$ is a solution of

$$m^2c^2\varphi - [[\varphi, \mathcal{P}_\mu]_\_, \mathcal{P}^\mu]_\_ = 0. \tag{5.5}$$

This is the *Klein-Gordon equation in momentum picture.* As a consequence of (4.12), this equation is valid in Heisenberg picture too, when it is also a corollary of the Klein-Gordon equation and the Heisenberg relation (4.28b).

The Euler-Lagrange equations (5.3) are not enough for determination of the field operators φ_i. This is due to the simple reason that in them enter also the components $\mathcal{P}_\mu$ of the (canonical) momentum operator (2.5), which are functions (functionals) of the field operators. Hence, a complete system of equations for the field operators should consists of (5.3) and an explicit connection between them and the momentum operator.

Since the densities of the conserved operators of a system are polynomial functions of the field operators and their partial derivatives in Heisenberg picture (for a polynomial Lagrangian of type (2.1)), in momentum picture they became polynomial functions of φ_i and $y_{j\nu} = \frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\nu]_\_$. When working in momentum picture, in view of (4.4), the following representations turn to be useful:

$$\mathcal{P}_\mu = \tilde{\mathcal{P}}_\mu = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \mathcal{T}_{0\mu} \circ \mathcal{U}(x, x_0)\, \mathrm{d}^3\boldsymbol{x} \tag{5.6}$$

$$\tilde{\mathcal{Q}} = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \mathcal{J}_0 \circ \mathcal{U}(x, x_0)\, \mathrm{d}^3\boldsymbol{x} \tag{5.7}$$

$$\tilde{\mathcal{L}}_{\mu\nu}(x) = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \{x_\mu \mathcal{T}^0{}_\nu - x_\nu \mathcal{T}^0{}_\mu\} \circ \mathcal{U}(x, x_0)\, \mathrm{d}^3\boldsymbol{x} \tag{5.8}$$

$$\tilde{\mathcal{S}}_{\mu\nu}(x) = \frac{1}{c} \int\limits_{x^0=\mathrm{const}} \mathcal{U}^{-1}(x, x_0) \circ \mathcal{S}^0_{\mu\nu} \circ \mathcal{U}(x, x_0)\, \mathrm{d}^3\boldsymbol{x}. \tag{5.9}$$

In particular, the combination of (5.6) and (5.3) (together with an explicit expression for the energy-momentum tensor $\mathcal{T}_{\mu\nu}$) provide a closed algebraic-functional system of equations for determination of the field operators φ_i in momentum picture. In fact, this is the *system of field equations in momentum picture.* Concrete types of such systems of field equations and their links with the (anti)commutation (and paracommutation) relations will be investigated elsewhere.

In principle, from (5.6)–(5.9) and the field equations (i.e. (5.6) and (5.3)) can be found the commutation relations between the conserved quantities and the momentum operator, i.e. $[\tilde{\mathcal{D}}, \tilde{\mathcal{P}}_\lambda]_\_$ with $\tilde{\mathcal{D}} = \tilde{\mathcal{P}}_\mu, \tilde{\mathcal{Q}}, \tilde{\mathcal{M}}_{\mu\nu}$. If one succeeds in computing $[\tilde{\mathcal{D}}, \tilde{\mathcal{P}}_\lambda]_\_$, one can calculate $[\tilde{\mathcal{D}}, \mathcal{U}(x, x_0)]_\_$ and, via (4.13), the operator $\mathcal{D} = \mathcal{P}_\mu, \mathcal{Q}, \mathcal{M}_{\mu\nu}$ in momentum picture. If it happens that (4.14) holds for $\tilde{\mathcal{A}} = \tilde{\mathcal{D}}$, then one can use simply the formula (4.16). In particular, this is the case if the commutators $[\tilde{\mathcal{D}}, \tilde{\mathcal{P}}_\lambda]_\_$ coincide with relations like (3.4) and (3.5) (see also [13, 5]).[5] For instance, if (3.5) holds, then (4.16) yields $\mathcal{Q} = \tilde{\mathcal{Q}}$, i.e. the charge operator remains one and the same in momentum and in Heisenberg pictures. Obviously, the last result holds for any operator commuting with the momentum operator.

A constant operator $\tilde{\mathcal{C}}$ in Heisenberg picture,

$$\partial_\mu \tilde{\mathcal{C}} = 0, \tag{5.10}$$

[5] In future work(s), it will be proved that, in fact, the so-calculated commutators $[\tilde{\mathcal{D}}, \tilde{\mathcal{P}}_\lambda]_\_$ reproduce similar relations, obtained from pure geometrical reasons in Heisenberg picture, at least for the most widely used Lagrangians. However, for the above purpose, one cannot use directly the last relations, except (3.4) in this case, because they are external to the Lagrangian formalism, so that they represent additional restriction to its consequences.

transforms in momentum picture into an operator $\mathcal{C}(x)$ such that

$$\partial_\mu \mathcal{C}(x) + \frac{1}{\mathrm{i}\hbar}[\mathcal{C}(x),\, \mathcal{P}_\mu]_{\_} = 0, \tag{5.11}$$

due to (4.9) and (4.12). In particular, the conserved quantities (e.g., the momentum, charge and angular momentum operators) are solutions of equation (5.11), i.e. a conserve operator need not to be a constant one in momentum picture, but it necessarily satisfies (5.11). Obviously, a constant operator $\tilde{\mathcal{C}}$ in Heisenberg picture is such in momentum picture if and only if it commutes with the momentum operator,

$$\partial_\mu \mathcal{C}(x) = 0 \iff [\mathcal{C}(x),\, \mathcal{P}_\mu]_{\_} = 0. \tag{5.12}$$

Such an operator, by virtue of (4.14) and (4.16), is one and the same in Heisenberg and momentum pictures,

$$\mathcal{C}(x) = \tilde{\mathcal{C}}. \tag{5.13}$$

In particular, the dynamical variables which are simultaneously measurable with the momentum, i.e. commuting with $\tilde{\mathcal{P}}_\mu$, remain constant in momentum picture and, hence, coincide with their values in Heisenberg one. Of course, such an operator is $\tilde{\mathcal{P}}_\mu = \mathcal{P}_\mu$, as we suppose the validity of (4.28a), and the charge operator $\tilde{\mathcal{Q}} = \mathcal{Q}$, if (3.5) holds.

Evidently, equation (5.11) is a 4-dimensional analogue of $\mathrm{i}\hbar\frac{\partial \mathcal{A}(t)}{\partial t} + [\mathcal{A}(t),\, \mathcal{H}(t)]_{\_} = 0$, which is a necessary and sufficient condition (in Schrödinger picture) for an observable $\mathcal{A}(t)$ to be an integral of motion of a quantum system with Hamiltonian $\mathcal{H}(t)$ in non-relativistic quantum mechanics [2, 15].

At the end, let us consider the Heisenberg relations (3.1)–(3.3) in momentum picture. As we said above, the first of them reduces to (4.19) in momentum picture and simply expresses the constantcy of the field operators φ_i. Since (3.2) has polynomial structure with respect to $\tilde{\varphi}_i(x)$, the transition to momentum picture preserves it, i.e. we have (see (4.4))

$$[\varphi_i,\, \mathcal{Q}]_{\_} = \varepsilon(\varphi_i) q_i \varphi_i \qquad \varepsilon(\varphi_i) = \varepsilon(\tilde{\varphi}_i). \tag{5.14}$$

At last, applying (4.4) to the both sides of (3.3) and taking into account (5.1), we obtain

$$[\varphi_i,\, \mathcal{M}_{\mu\nu}(x, x_0)]_{\_} = x_\mu[\varphi_i,\, \mathcal{P}_\nu]_{\_} - x_\nu[\varphi_i,\, \mathcal{P}_\mu]_{\_} + \mathrm{i}\hbar I^j_{i\mu\nu}\varphi_j. \tag{5.15}$$

However, in a pure Lagrangian approach, to which (5.14) and (5.15) are external restrictions, one is not allowed to apply (5.14) and (5.15) unless these equations are explicitly proved for the operators $\mathcal{M}_{\mu\nu}$ and $\mathcal{P}_\mu$ given via (2.5)–(2.8) and (4.4).

6. On the momentum representation and particle interpretation

An important role in quantum field theory plays the so-called *momentum representation* (in Heisenberg picture) [3, 13, 11]. Its essence is in the replacement of the field operators $\tilde{\varphi}_i(x)$ with their Fourier images $\tilde{\phi}_i(k)$, both connected by the Fourier transform ($kx := k_\mu x^\mu$)[6]

$$\tilde{\varphi}_i(x) = \int \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}kx}\, \tilde{\phi}_i(k)\, \mathrm{d}^4 k, \tag{6.1}$$

and then the representation of the field equations, dynamical variables, etc. in terms of $\tilde{\phi}_i(k)$.

[6] For brevity, we omit the inessential for us factor, equal to a power of 2π, in the r.h.s. of (6.1).

Applying the general rule (4.4) to (6.1), we see that the analogue of $\tilde{\phi}_i(k)$ in momentum picture is the operator

$$\phi_i(k) := \mathrm{e}^{-\frac{1}{\mathrm{i}\hbar}kx}\,\mathcal{U}(x,x_0)\circ\tilde{\phi}_i(k)\circ\mathcal{U}^{-1}(x,x_0), \tag{6.2}$$

which is independent of x, depends generally on x_0 and is such that

$$\varphi_i = \int \phi_i(k)\,\mathrm{d}^4k. \tag{6.3}$$

A field theory in terms of the operators $\phi_i(k)$ will be said to be in the *momentum representation* in momentum picture.

The Heisenberg relation (4.28b) in momentum representation, evidently, reads

$$[\tilde{\phi}_i(k),\tilde{\mathcal{P}}_\mu]_{\_} = -k_\mu\,\tilde{\phi}_i(k) \qquad [\phi_i(k),\tilde{\mathcal{P}}_\mu]_{\_} = -k_m\,\phi_i(k) \tag{6.4}$$

in Heisenberg and momentum picture, respectively.[7] Consider a state vector $\tilde{\mathcal{X}}_p$ with fixed 4-momentum $p=(p_0,\dots,p_3)$, i.e. for which

$$\tilde{\mathcal{P}}_\mu(\tilde{\mathcal{X}}_p) = p_\mu\,\tilde{\mathcal{X}}_p \qquad \mathcal{P}_\mu(\mathcal{X}_p) = p_\mu\,\mathcal{X}_p. \tag{6.5}$$

Combining these equations with (6.4), we get

$$\tilde{\mathcal{P}}_\mu\big(\tilde{\phi}_i(k)(\tilde{\mathcal{X}}_p))\big) = (p_\mu+k_\mu)\,\tilde{\phi}_i(k)(\tilde{\mathcal{X}}_p)) \qquad \mathcal{P}_\mu\big(\phi_i(k)(\mathcal{X}_p))\big) = (p_\mu+k_\mu)\,\phi_i(k)(\mathcal{X}_p)). \tag{6.6}$$

So, the operators $\tilde{\phi}_i(k)$ and $\phi_i(k)$ increase the state's 4-momentum p_μ by k_μ. If it happens that $k_0\ge 0$, we can say that these operators create a particle with 4-momentum $(\sqrt{k^2-\boldsymbol{k}^2},\boldsymbol{k})$. (Notice $k^2=k_0^2+\boldsymbol{k}^2$, $\boldsymbol{k}:=(k^1,k^2,k^3)$, need not to be a constant in the general case, so the mass $m:=\frac{1}{c}\sqrt{k^2}$ is, generally, momentum-dependent.) One can introduce the creation/annihilation operators by

$$\tilde{\phi}_i^{\pm}(k) := \begin{cases} \tilde{\phi}_i(\pm k) & \text{for } k_0\ge 0\\ \frac{1}{2}\,\tilde{\phi}_i(\pm k) & \text{for } k_0 = 0\\ 0 & \text{for } k_0<0 \end{cases} \qquad \phi_i^{\pm}(k) := \begin{cases} \phi_i(\pm k) & \text{for } k_0\ge 0\\ \frac{1}{2}\,\phi_i(\pm k) & \text{for } k_0 = 0\\ 0 & \text{for } k_0<0 \end{cases}. \tag{6.7}$$

In terms of them, equations (6.6) take the form

$$\tilde{\mathcal{P}}_\mu\big(\tilde{\phi}_i^{\pm}(k)(\tilde{\mathcal{X}}_p))\big) = (p_\mu\pm k_\mu)\,\tilde{\phi}_i(k)(\tilde{\mathcal{X}}_p)) \qquad \mathcal{P}_\mu\big(\phi_i^{\pm}(k)(\mathcal{X}_p))\big) = (p_\mu\pm k_\mu)\,\phi_i(k)(\mathcal{X}_p)). \tag{6.8}$$

Thus, if $k_0\ge 0$, we can interpret $\tilde{\phi}_i^{+}(k)$ and $\phi_i^{+}(k)$ (resp. $\tilde{\phi}_i^{-}(k)$ and $\phi_i^{-}(k)$) as operators creating (resp. annihilating) a particle with 4--momentum k_μ.

If the relations (3.2) (resp. (3.3)) hold, similar considerations are (resp. partially) valid with respect to state vectors with fixed charge (resp. total angular momentum).

As we see, the description of a quantum field theory in momentum representation is quite similar in Heisenberg picture, via the operators $\tilde{\phi}_i(k)$, and in momentum picture, via the operators $\phi_i(k)$. This similarity will be investigated deeper on concrete examples in forthcoming paper(s). The particular form of the operators $\tilde{\phi}_i(k)$ and $\phi_i(k)$ can be found by solving the field equations, respectively (2.2) and (5.3), in momentum representation, but the analysis of the so-arising equations is out of the subject of the present work.

[7] The equations (6.4) are a particular realization of a general rule, according to which any linear combination, possibly with operator coefficients, of $\tilde{\varphi}_i(x)$ and their partial derivatives (up to a finite order) transforms into a polynomial in k_μ, the coefficients of which are proportional to $\tilde{\phi}_i(k)$. By virtue of (6.2), the same result holds in terms of $\phi_i(k)$ instead of $\tilde{\phi}_i(k)$, i.e. in momentum picture.

7. The momentum picture as 4-dimensional analogue of the Schrödinger one

We have introduced the momentum picture and explored some its aspects on the base of the Heisenberg one, i.e. the latter picture was taken as a ground on which the former one was defined and investigated; in particular, the conditions (4.28) turn to be important from this view-point. At that point, a question arises: can the momentum picture be defined independently and to be taken as a base from which the Heisenberg one to be deduced? Below is presented a partial solutions of that problem for theories in which the equations (4.28) hold.

First of all, it should be decided which properties of the momentum picture, considered until now, characterize it in a more or less unique way and then they or part of them to be incorporated in a suitable (axiomatic) definition of momentum picture. As a guiding idea, we shall follow the understanding that the momentum picture is (or should be) a 4-dimensional analogue of the Schrödinger picture in non-relativistic quantum mechanics. Recall, [2,15,16], the latter is defined as a representation of quantum mechanics in which: (i) the operators, corresponding to the dynamical variables, are time-independent; (ii) these operators are taken as predefined (granted) in an appropriate way; and (iii) the wavefunctions ψ are, generally, time-dependent and satisfy the Schrödinger equation

$$\frac{\partial \psi}{\partial t} = \frac{1}{\mathrm{i}\hbar} \mathcal{H}(\psi), \tag{7.1}$$

with $\mathcal{H}$ being the system's Hamiltonian acting on the system's Hilbert space of states. A 4-dimensional generalization of (i)–(iii), adapted for the needs of quantum field theory, will result in an independent definition of the momentum picture. Since in that theory the operators of the dynamical variables are constructed form the field operators φ_i, the latter should be used for the former ones when the generalization mentioned is carried out. Besides, the field operators satisfy some equations, which have no analogues in quantum mechanics, which indicates to a nontrivial generalization of item (ii) above.

Following these ideas, we define the *momentum picture* of quantum field theory as its representation in which:

(a) The field operators φ_i are spacetime-independent,

$$\partial_\mu(\varphi_i) = 0. \tag{7.2}$$

(b) The state vectors χ are generally spacetime-dependent and satisfy the following first order system of partial differential equations

$$\partial_\mu(\chi) = \frac{1}{\mathrm{i}\hbar} \mathcal{P}_\mu(\chi), \tag{7.3}$$

where $\mathcal{P}_\mu$ are the components of the system's momentum operator (constructed according to point (c) below – see (7.8)). If $\chi_0 \in \mathcal{F}$ and $x_0 \in M$ are fixed, the system (7.3) is supposed to have a unique solution satisfying the initial condition

$$\chi|_{x=x_0} = \chi_0. \tag{7.4}$$

(c) If $\tilde{\mathcal{D}}(\tilde{\varphi}_i, \partial_\mu \tilde{\varphi}_j)$ is the density current of a dynamical variable in (ordinary) Heisenberg picture, which is supposed to be polynomial or convergent power series in $\tilde{\varphi}_i$ and $\partial_\mu \tilde{\varphi}_j$, then this quantity in momentum picture is defined to be

$$\mathcal{D} = \mathcal{D}(\varphi_i) := \tilde{\mathcal{D}}\big(\varphi_i, \frac{1}{\mathrm{i}\hbar}[\varphi_j, \mathcal{P}_\mu]_{\_}\big). \tag{7.5}$$

The corresponding spacetime conserved operator is defined as

$$\mathsf{D} := \frac{1}{c}\int_{x_0=\text{const}} \mathcal{U}^{-1}(x,x_0) \circ \mathcal{D}(\varphi_i) \circ \mathcal{U}(x,x_0)\,\mathrm{d}^3\boldsymbol{x}, \tag{7.6}$$

where $\mathcal{U}(x,x_0)$ is the evolution operator for (7.3)–(7.4), i.e. the unique solution of the initial-value problem

$$\frac{\partial\mathcal{U}(x,x_0)}{\partial x^\mu} = \frac{1}{\mathrm{i}\hbar}\mathcal{P}_\mu \circ \mathcal{U}(x,x_0) \tag{7.7a}$$
$$\mathcal{U}(x_0,x_0) = \mathsf{id}_{\mathcal{F}} \tag{7.7b}$$

with $\mathcal{P}_\mu$ corresponding to (7.6) with the energy-momentum tensor $\mathcal{T}_{\mu\nu}$ for $\mathcal{D}$,

$$\mathcal{P}_\mu := \frac{1}{c}\int_{x_0=\text{const}} \mathcal{U}^{-1(x,x_0)} \circ \mathcal{T}_{0\mu}(\varphi_i) \circ \mathcal{U}(x,x_0)\,\mathrm{d}^3\boldsymbol{x}. \tag{7.8}$$

(d) The field operators φ_i are solutions of the (algebraic) field equations, which (in the most cases) are identified with the Euler-Lagrange equations

$$\Big\{\frac{\partial\,\tilde{\mathcal{L}}(\varphi_j,y_{l\nu})}{\partial\varphi_i} - \frac{1}{\mathrm{i}\hbar}\Big[\frac{\partial\,\tilde{\mathcal{L}}(\varphi_j,y_{l\nu})}{y_{i\mu}},\,\mathcal{P}_\mu\Big]_{\_}\Big\}\Big|_{y_{j\nu}=\frac{1}{\mathrm{i}\hbar}[\varphi_j,\,\mathcal{P}_\nu]_{\_}} = 0, \tag{7.9}$$

with $\tilde{\mathcal{L}}(\varphi_j,\frac{1}{\mathrm{i}\hbar}[\varphi_j,\,\mathcal{P}_\nu]_{\_})$ being the system's Lagrangian (in momentum picture, defined according to (7.5)).

A number or comments on the conditions (a)–(d) are in order.

The transition from momentum to Heisenberg picture is provided by the inversion of (4.3) and (4.3) with $\mathcal{U}(x,x_0)$ given via (7.7), i.e.

$$\mathcal{X} \mapsto \tilde{\mathcal{X}} = \mathcal{U}^{-1}(x,x_0)(\,\tilde{\mathcal{X}}(x)) \tag{7.10}$$
$$\mathcal{A}(x) \mapsto \tilde{\mathcal{A}}(x) = \mathcal{U}^{-1}(x,x_0) \circ (\,\mathcal{A}(x)) \circ \mathcal{U}(x,x_0). \tag{7.11}$$

Since (7.7) implies

$$\mathcal{H}_\mu(x,x_0) = \frac{1}{\mathrm{i}\hbar}\mathcal{P}_\mu \tag{7.12}$$

for the quantities (4.7), the replacement (4.9) is valid. In particular, we have

$$\partial_\mu\,\tilde{\varphi}_i \mapsto y_{j\mu} = \frac{1}{\mathrm{i}\hbar}[\varphi_j,\,\mathcal{P}_\mu]_{\_}\ , \tag{7.13}$$

by virtue of (7.2), which justifies the definition (7.5) and the equation (7.9). The Heisenberg relations (4.28b) follow from this replacement:

$$[\,\tilde{\varphi}_i(x),\,\tilde{\mathcal{P}}_\mu]_{\_} = \mathcal{U}^{-1}(x,x_0) \circ [\,\varphi_i,\,\mathcal{P}_\mu]_{\_} \circ \mathcal{U}(x,x_0) = \mathrm{i}\hbar\partial_\mu(\,\tilde{\varphi}_i).$$

Since the integrability conditions for (7.3) are

$$\begin{aligned}0 = \partial_\nu \circ \partial_\mu(\chi) - \partial_\mu \circ \partial_\nu(\chi) &= \frac{1}{\mathrm{i}\hbar}\big\{\partial_\nu(\,\mathcal{P}_\mu(\chi)) - \partial_\mu(\,\mathcal{P}_\nu(\chi))\big\}\\ &= \frac{1}{\mathrm{i}\hbar}\big\{(\partial_\nu(\,\mathcal{P}_\mu) - \partial_\mu(\,\mathcal{P}_\nu))(\chi) + \mathcal{P}_\mu(\partial_\nu(\chi)) - \,\mathcal{P}_\nu(\partial_\mu(\chi))\big\},\end{aligned}$$

where (7.3) was applied, the existence of a unique solution of (7.3)–(7.4) implies (use (7.2) again; cf. footnote 3)

$$\partial_\nu(\,\mathcal{P}_\mu) - \partial_\mu(\,\mathcal{P}_\nu)) + \frac{1}{\mathrm{i}\hbar}[\,\mathcal{P}_\mu,\,\mathcal{P}_\nu]_{\_} = 0. \tag{7.14}$$

As $\partial_\nu \tilde{\mathcal{P}}_\mu = 0$, due to the conservation of $\tilde{\mathcal{P}}_\mu$, the replacement (4.6), with $\mathcal{P}_\nu$ for $\mathcal{A}(x)$, together with (7.12) entails $\partial_\mu(\mathcal{P}_\nu)) + \frac{1}{\mathrm{i}\hbar}[\mathcal{P}_\nu, \mathcal{P}_\mu]_{\_} = 0$, which, when inserted into (7.14), gives

$$\partial_\nu(\mathcal{P}_\mu) = 0. \tag{7.15}$$

The substitution of (7.15) into (7.14) results in

$$[\mathcal{P}_\mu, \mathcal{P}_\nu]_{\_} = 0, \tag{7.16}$$

which immediately implies (4.28a).

As a result of (7.16) and (7.7), we obtain

$$\mathcal{U}(x, x_0) = \mathrm{e}^{\frac{1}{\mathrm{i}\hbar}(x^\mu - x_0^\mu)\mathcal{P}_\mu}, \tag{7.17}$$

so that

$$[\mathcal{U}(x, x_0), \mathcal{P}_\mu]_{\_} = [\mathcal{U}^{-1}(x, x_0), \mathcal{P}_\mu]_{\_} = 0 \tag{7.18}$$

and, consequently

$$\tilde{\mathcal{P}}_\mu = \mathcal{U}^{-1}(x, x_0) \circ \mathcal{P}_\mu \circ \mathcal{U}(x, x_0) = \mathcal{P}_\mu, \tag{7.19}$$

which implies the coincidence of the evolution operators given by (4.1) and (7.7). The last conclusion leads to the identification of the momentum picture defined via the conditions (a)–(d) above and by (4.3), (4.4) and (4.28) in Sect. 4.

What regards the conditions (c) and (d) in the definition of the momentum picture, they have no analogues in quantum mechanics. Indeed, equations (7.5)–(7.9) form a closed system for determination of the field operators (via the so-called creation and annihilation operators) and, correspondingly, they provide a method for obtaining explicit forms of the dynamical variables (via the same operators). On the contrary, in quantum mechanics there is no procedure for determination of the operators of the dynamical variables and they are defined by reasons external to this theory.

Thus, we see that a straightforward generalization of the Schrödinger picture in quantum mechanics to the momentum picture in quantum field theory (expressed first of all by (7.2) and (7.3)) is possible if and only if the equations (4.28) are valid for the system considered.

8. Conclusion

In the present paper, we have summarized, analyzed and developed the momentum picture of motion in (Lagrangian) quantum field theory, introduced in [1]. As it was shown, this picture is (expected to be) useful when the conditions (4.28) are valid in (or compatible with) the theory one investigates. If this is the case, the momentum picture has properties that allow one to call it a '4-dimensional Schrödinger picture' as the field operators (and functions which are polynomial in them and their derivatives) in it became spacetime-constant operators and the state vectors have a simple, exponential, dependence on the spacetime coordinates/points. This situation is similar to the one in quantum mechanics in Schrödinger picture, when time-independent Hamiltonians are employed [2], the time replacing the spacetime coordinates in our case.

As we said in Sect 4, there are evidences that the conditions (4.28) should be a part of the basic postulates of quantum field theory (see also [13, § 68]). In the ordinary field theory, based on the Lagrangian formalism to which (anti)commutation relations are added as additional conditions [3,4,11], the validity of (4.28) is questionable and should be checked for any particular Lagrangian [13]. The cause for this situation lies in the fact that (4.28) and the (anti)commutation relations are additional to the Lagrangian formalism and their compatibility is a problem whose solution is not obvious. The solution of that problem is

known to be positive for a lot of particular Lagrangians [13], but, in the general case, it seems not to be explored. For these reasons, one may try to 'invert' the situation, i.e. to consider a Lagrangian formalism, to which the conditions (4.28) are imposed as subsidiary restrictions, and then to try to find (anti)commutation relations that are consistent with the so-arising scheme. We intend to realize this program in forthcoming papers, in which it will be demonstrated that the proposed method reproduces most of the known results, reveals ways for their generalizations at different stages of the theory, and also gives new results, such as a (second) quantization of electromagnetic field in Lorentz gauge, imposed directly on the field's operator-valued potentials, and a 'natural' derivation of the paracommutation relations.

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